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Note

Use of Olver's Algorithm to Evaluate Certain Definite Integrals of Plasma Physics Involving Chebyshev Polynomials

W. L. SADOWSKI AND D. W. LOZIER

Institute for Basic Standards, National Bureau of Standards, Washington DC 20234

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INTRODUCTION

Three methods were used to evaluate certain integrals involving Chebyshev polynomials. The paper shows that a seemingly simple computational problem may be intractable unless a suitable algorithm is chosen. In this case the simplest conceivable method fails badly and a more sophisticated method has severe limitations on the significance that can be obtained. The third method is based on a three-term inhomogeneous recursion. Asymptotic analysis shows that there is a rapidly increasing solution and a rapidly decreasing solution, while the desired solution decreases slowly. Therefore, neither forward nor backward recurrence is stable for this method. We will demonstrate the application of the powerful algorithm developed by Olver [1] to obtain the solution.

The problem arose in calculating the electric field and kinetic energy for an inhomogeneous periodic nonlinear plasma by means of a Fourier–Chebyshev expansion [2, 3, 4].

Chebyshev polynomials used in the expansion are those of the first kind [5], viz.,

$$T_n(v) = \cos(n \cos^{-1} v)$$

for $-1 \leqslant v \leqslant 1$ and nonnegative integer n. The Chebyshev polynomials form an orthogonal system on $[-1, 1]$ with weight function $(1 - v^2)^{-1/2}$. Note that $T_n(v)$ is an even polynomial if n is even, and odd if n is odd.

The problem reduces to the calculation of the following sequence of integrals:

$$S_n = \int_{-1}^{1} T_{2n}(v)\, e^{-s^2 v^2/2}\, dv, \qquad n = 0, 1, 2, \ldots . \tag{1}$$

Since $|T_{2n}(v)| \leqslant 1$ the integrals (1) are bounded by S_0. Furthermore, they converge to zero as n tends to infinity, a fact that follows easily from the Riemann–Lebesgue lemma after setting $v = \cos\theta$.

Method I. The first approach that comes to mind is term-by-term integration of the expressions obtained by representing the Chebyshev polynomials in powers of v. Then

$$S_n = \sum_{k=0}^{n} (-1)^{n+k} a_k{}^n V_k , \tag{2}$$

where

$$V_k = \int_{-1}^{1} v^{2k} e^{-s^2v^2/2}\, dv \tag{3}$$

and $(-1)^{n+k} a_k{}^n$ is the coefficient of v^{2k} in $T_{2n}(v)$. Values of V_k can be obtained by backward recurrence after an integration by parts. S_n calculated this way are given in Tables I and II under Method I.

Method II. The second approach is based on expanding the exponential $e^{-s^2v^2/2}$ in Chebyshev polynomials:

$$e^{-s^2v^2/2} = \tfrac{1}{2}c_0 + \sum_{k=1}^{\infty} (-1)^k c_k T_{2k}(v). \tag{4}$$

It can be shown that

$$c_k = 2e^{-s^2/4} I_k(s^2/4), \tag{5}$$

where I_k are the modified Bessel functions [6]. By using identities for products of T_n to represent the integrand we obtain a sequence of approximations to S_n

$$S_n{}^K = \tfrac{1}{2}c_0 b_0{}^n + \sum_{k=1}^{K} (-1)^k b_k{}^n c_k , \tag{6}$$

where $b_k{}^n = -1/(4(n+k)^2 - 1) - 1/(4(n-k)^2 - 1)$. The approximations (6) converge quite rapidly to S_n as $K \to \infty$. Results of calculating S_n on the basis of (6) are shown in Tables I and II under Method II.

Method III (*Olver's Algorithm*). The most satisfactory approach is based on an inhomogeneous second-order linear difference equation obtained by integrating (1) by parts. Using identities on Chebyshev polynomials [5], we obtain from (1)

$$\frac{s^2(2n-1)}{8} S_{n+1} - \frac{s^2 + 2(4n^2-1)}{4} S_n - \frac{s^2(2n+1)}{8} S_{n-1} = e^{-s^2/2} \tag{7}$$

after some rearrangement.

The asymptotic nature of the solutions of Eq. (7) can be expected to be similar to the solutions of the equation

$$(s^2/4)\,S_{n+1} - 2nS_n - (s^2/4)\,S_{n-1} = e^{-s^2/2}/n, \tag{8a}$$

which is obtained by neglecting terms of relative order $1/n$ in the coefficients. In a similar way, the solutions of (8a) are approximated by $S_n = Q_n/n$ where Q_n satisfies

$$Q_{n+1} - 2\alpha n Q_n - Q_{n-1} = \alpha e^{-2/\alpha}, \tag{8b}$$

where $\alpha = 4/s^2$.

The solutions of the homogeneous part of (8b) are the Bessel functions $(-1)^n I_n(1/\alpha)$ and $K_n(1/\alpha)$. For large n and fixed α, these functions have the following asymptotic forms [5]:

$$I_n(1/\alpha) \sim 1/(2\alpha)^n\, n! \tag{9}$$

$$K_n(1/\alpha) \sim \tfrac{1}{2}(2\alpha)^n\,(n-1)! \tag{10}$$

As will be shown below, the solution we are trying to calculate lies between the solutions (9) and (10). Consequently, once a small component of (10) is present in the solution obtained by forward recursion, the calculation will be eventually dominated by (10). The same holds true for (9) when backward recursion is used. The backward recursion will be dominated by the most rapidly growing solution, in this case (9).

We can see where the desired solution lies by looking at the asymptotic behavior of the particular solution of (8b) given below:

$$\begin{aligned}\bar{Q}_n = {} & -e^{-2/\alpha} K_n(1/\alpha) \sum_{k=n}^{\infty} I_k(1/\alpha) \\ & - e^{-2/\alpha}(-1)^n I_n(1/\alpha) \sum_{k=0}^{n-1} (-1)^k K_k(1/\alpha).\end{aligned} \tag{11}$$

This solution can be obtained by the analog to the method known in differential equation theory as variation of parameters. It can be verified by substitution. For large n, retaining dominant terms in (11) we obtain

$$\bar{Q}_n \sim -e^{-2/\alpha}(1/2n) + e^{-2/\alpha}[1/4\alpha n(n-1)] \sim -e^{-2/\alpha}(1/2n). \tag{12}$$

Since $S_n = Q_n/n$, all of the analysis on Q_n applies to S_n as well. Since it is clear from (12) that neither the forward recursion dominated by (10), nor the backward recursion dominated by (9), would yield $1/n^2$ type behavior of S_n, a more powerful solution technique is required. Such a technique is the method developed by Olver [1].

The method is applicable to an arbitrary inhomogeneous linear difference equation of second order provided only that there exist two complementary solutions such that one grows asymptotically more rapidly than the other, i.e., their ratio tends to zero, and that the desired particular solution is dominated by the more rapidly growing complementary solution. Our asymptotic analysis indicated that these conditions are met for the difference Eq. (7).

Essentially, the method makes it possible to solve (7) as a boundary value problem rather than as an initial value problem. The first boundary condition is the initial point S_0, which must be calculated separately. The second boundary condition is obtained by setting S_n equal to zero for some integer n. The integer n is determined automatically in such a way that truncation errors in the algorithm will lie within a preassigned tolerance. The computation proceeds by simple recurrences that are numerically stable.

Comparison of Results. Since Olver's algorithm gives results to within a preassigned error, these will be used as a standard for comparison of the methods.

Table I gives results obtained by Methods I, II, and III for $s = 6$. As can be seen, Method I is unstable and by the time $n = 24$ is reached, all significance has been lost, while Method II retains eleven or more correct significant figures throughout the range given.

Table II gives results for $s = 10$. Here Method I gives results with no correct significant figures at $n = 36$, while Method II still has eleven correct significant figures at this point. In the asymptotic region, however, for values of $n > 61$, neither Method I nor Method II have any significant figures.

The qualitative behavior of results obtained by Methods I and II is explained in the following way. The onset of severe cancellation errors in the sum (2) of Method I occurs at higher values of n for increasing s because V_k goes to zero more rapidly with increasing k as s is increased. The behavior of the results of Method II as a function of s is opposite from that of Method I. As can be seen from (1) the integrals decrease more rapidly with increasing n as s is increased. But from (6) one can see that Method II yields results on the basis of cancellation of terms which are of the order of $1/n^2$ at most.

Consequently, in a double precision calculation which carries eighteen figures no function value can be computed by Method II that is smaller than $10^{-18}/n^2$. This is why for $s = 6$ (Table I), where values in the asymptotic region are of the order of 10^{-11}, Method II can yield eleven significant figures but for $s = 10$ (Table II), where values in the asymptotic region are of the order of 10^{-25}, it can yield no significance.

Finally, ratios of values obtained by Method III do indeed show the $1/n^2$

TABLE I

Comparison of the Three Methods for $s = 6$[a]

	METHOD I	METHOD II	METHOD III
0	.100000000000000000+001	.100000000000000000+001	.100000000000000000+001
3	-.600823050968318077+000	-.600823050968318074+000	-.600823050968318075+000
6	.138037871203706905+000	.138037871203706910+000	.138037871203706910+000
9	-.137587807838599484-001	-.137587807838599644-001	-.137587807838599643-001
12	.677485093416898589-003	.677485093416780049-003	.677485093416780040-003
15	-.183682147328334544-004	-.183682147367764981-004	-.183682147367764981-004
18	.298608819893075861-006	.298609156458961616-006	.298609156458961552-006
21	-.316919113174662925-008	-.315259883712369007-008	-.315259883712477334-008
24	.248019205173477530-008	-.833584909587225002-011	-.833584909647467205-011
27	-.704632839187979698-006	-.242452604981588188-010	-.242452604981042025-010
30	-.901687890291213989-004	-.196775748929573088-010	-.196775748936271088-010
33	.162205696105957031-001	-.163422139227326907-010	-.163422139231636394-010
36	.117095947265625000+001	-.137834497219897597-010	-.137834497222359413-010
41	-.728703124999999999+003	-.106750436376746913-010	-.106750436383238470-010
46	.140517120000000000+008	-.850745151399045327-011	-.850745151383484165-011
51	-.100515688448000000+012	-.693702864059862927-011	-.693702864085216054-011
56	.124592783758458880+017	-.576347447556942451-011	-.576347447581590596-011
61	.480775228160305686+022	-.486377294542877643-011	-.486377294546637896-011
66	.277634301698929643+028	-.415904532979430408-011	-.415904533000723719-011
71	.232856147102766833+034	-.359685333794932457-011	-.359685333807200350-011
76	.276826073139590849+040	-.314124877212290769-011	-.314124877215181108-011
81	.456065710681834452+046	-.276692789738979588-011	-.276692789743996392-011
86	.102078333698590947+053	-.245566548276447121-011	-.245566548277336263-011

[a] The values have been normalized such that $S_0 = 1$.

TABLE II

Comparison of the Three Methods for $s = 10$[a]

	METHOD I	METHOD II	METHOD III
0	.10000000000000000+001	.10000000000000000+001	.10000000000000000+001
3	-.83392000000000002+000	-.83392000000000001+000	-.83392000000000000+000
6	.48493828096000000+000	.48493828096000000+000	.48493828096000002+000
9	-.198194921301606402+000	-.198194921301606400+000	-.198194921301606402+000
12	.576159090815908399-001	.576159090815908449-001	.576159090815908452-001
15	-.120914270781679555-001	-.120914270781679724-001	-.120914270781679725-001
18	.186224707091716316-002	.186224707091733611-002	.186224707091733613-002
21	-.214096829919864563-003	-.214096829921639743-003	-.214096829921639748-003
24	.186857097111269955-004	.186857097155317851-004	.186857097155317862-004
27	-.125824116477974455-005	-.125824122329365860-005	-.125824122329365958-005
30	.663729041282398128-007	.663740254860061164-007	.663740254860058351-007
33	-.271634248557792168-008	-.278213833656243841-008	-.278213833656292118-008
36	-.382935781001458864-008	.938857758539481341-010	.938857758532630587-010
41	-.587656955985949025-006	-.211061116650250380-012	-.211061117132297887-012
46	.428070219641085714-003	.282247470792032632-015	.282247600241896799-015
51	-.569626535754650831+000	-.233945757777112542-018	-.2343920723222371471-018
56	.187632228064537048+004	.407946758619428995-021	.1253447779143757851-021
61	-.395138110424804687+007	.375940912012943198-022	-.146095999405692705-024
66	-.101961371911250000+011	.280992977488044813-021	-.868364784058799348-025
71	.701462890771199998+014	.123102886316177929-021	-.752151587276050852-025
76	-.426219683771056127+018	.712614856617248191-022	-.657641801264606786-025
81	.112928966361693867+022	.582694512877966022-022	-.579830406024624147-025
86	.112152992557094662+025	.669980538318062581-022	-.515014488510612466-025

[a] The values have been normalized such that $S_0 = 1$.

behavior in the asymptotic region as predicted by the analysis. In fact, the agreement is good to three figures. We discuss ratios here because the computer output is such that S_0 in (1) is normalized to unity.

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